NONNEGATIVE GRASSMAN CHAMBERS ARE BALLS

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§0 Introduction

Classically, the notion of total positivity referred to matrices all of whose minors had positive determinants. Lusztig generalized this notion substantially ([L1],[L2],[L3]) introducing the nonnegative part of an arbitrary reductive group, as well as the nonnegative part of a flag variety. Lusztig proved that the latter is always contractible and it has been conjectured to always be homeomorphic to a closed ball. Some work in this direction may be found in [W1],[W2].

However, even the case of Grassmannians remained open. In this paper, we present an elementary proof that the nonnegative part of a Grassmannian is homeomorphic to a ball.

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§1 Multilinear algebra

In this section, we record two useful lemmas in multilinear algebra. We work on $\mathbb{R}^n$ and fix a basis $e_1, \dots, e_n$. We fix an inner product for which this basis is orthonormal. For any subset $A \subset \{1, \dots, n\}$ with $\#(A) = k$, we write

$$e_A = e_{a_1} \wedge \cdots \wedge e_{a_k},$$

where $a_1 < \cdots < a_k$ are the elements of A arranged from least to greatest. Clearly

$$e_A \in \Lambda^k(\mathbb{R}^n),$$

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and the collection $\{e_A\}$ form an orthonormal basis under the induced inner product. Whenever $\omega \in \Lambda^k(\mathbb{R}^n)$, we say that ω is *decomposable* provided that

$$\omega = v_1 \wedge \cdots \wedge v_k, \tag{1.1}$$

with $v_1, \ldots, v_k \in \mathbb{R}^n$. (Most authors refer to this condition as totally decomposable.) We write

$$\omega = \sum_A \omega_A e_A.$$

We say that ω is *normalized* if

$$\sum_A \omega_A = 1.$$

We say that ω is *positive* (resp. *nonnegative*) if each component ω_A is positive (resp. nonnegative.) The set of normalized, decomposable, nonnegative elements of $\Lambda^k(\mathbb{R}^n)$ is in one to one correspondence with the nonnegative elements of the Grassmannian $G(k,n)$ of k planes containing the origin in $\mathbb{R}^n$. Here with $v_1, \ldots, v_k$ as in (1.1), the k-vector ω corresponds to the k-plane spanned by $v_1, \ldots, v_k$. This one-to-one correspondence is a homeomorphism. If $j \leq k$ and $\omega \in \Lambda^k(\mathbb{R}^n)$, while $\eta \in \Lambda^j(\mathbb{R}^n)$, we say $\eta \subset \omega$ provided both η and ω are decomposable and the j-plane corresponding to η is contained in the k-plane corresponding to ω.

Therefore nonnegative decomposable elements of $\Lambda^k(\mathbb{R}^n)$ shall be our object of study. We prove two lemmas.

Lemma 1.1. *Let ω be a nonnegative, decomposable, normalized element of $\Lambda^k(\mathbb{R}^n)$. Then there is $\eta \in \Lambda^{k-1}(\mathbb{R}^n)$, nonnegative and nonzero with $\eta \subset \omega$. If ω is positive, then η may be chosen to be positive.*

Proof. To prove the first claim let

$$\omega = \sum_A \omega_A e_A,$$

be nonnegative and decomposable. Let j be the smallest number so that there exists A with ω_A nonzero and $j \in A$. (The k-vector ω cannot be zero since it is normalized.) Then by row reduction, we can write

$$\omega = (e_j + v_1) \wedge v_2 \wedge \cdots \wedge v_k,$$

where none of the v's has any component of e_l for $l \leq j$. Then the components of

$$e_j \wedge v_2 \wedge \cdots \wedge v_k,$$

must be nonnegative and we can set

$$\eta = v_2 \wedge \cdots \wedge v_k.$$

The second claim is a little more difficult. We fix $\epsilon > 0$ to be specified later. We proceed by induction. The claim is obvious for $n = k$ (and by duality for $k = 0$.) Now we assume it is true with k replaced by $k-1$ and with n replaced by $n-1$. We have

$$\omega = \sum_A \omega_A e_A,$$

with all the ω_A strictly positive. As before, we can rewrite

$$\omega = (e_1 + v_1) \wedge v_2 \cdots \wedge v_k,$$

where $v_1, \ldots v_k$ do not involve e_1.

We know that $v_2 \wedge \cdots \wedge v_k$ is positive when viewed as a $k-1$ vector in $\mathbb{R}^{n-1}$. Thus we may write

$$v_2 \wedge \cdots \wedge v_k = w_2 \wedge \cdots \wedge w_k,$$

with

$$w_3 \wedge \cdots \wedge w_k,$$

positive when viewed as a $k-2$ vector on $\mathbb{R}^{n-1}$. (Here we have used the induction hypothesis.) Next we observe that we can write $\epsilon\omega$ in the following peculiar way:

$$\epsilon\omega = (e_1 + v_1) \wedge (\epsilon w_2 + w_3) \wedge (-\epsilon^2(e_1 + v_1) + w_3) \wedge w_4 \cdots \wedge w_k.$$

Now we consider

$$\eta_\epsilon = (\epsilon w_2 + w_3) \wedge (-\epsilon^2(e_1 + v_1) + w_3) \wedge w_4 \cdots \wedge w_k.$$

We observe that the terms involving e_1 are $\epsilon^2 v_1 \wedge w_3 \cdots \wedge w_k + O(\epsilon^3)$ and the terms not involving e_1 are $\epsilon w_2 \wedge \cdots \wedge w_k + O(\epsilon^2)$. Therefore letting ϵ be sufficiently small, we see that η_ϵ is positive. But by our construction, for any ϵ, we have $\eta_\epsilon \subset \omega$.

Notice this proof only worked for $k \geq 3$. A minor modification takes care of the case $k = 2$. Then we write $\omega = (e_1 + v_1) \wedge v_2$. We set $\eta_\epsilon = \epsilon(e_1 + v_1) + v_2$. $\square$

Now we state the second lemma.

Lemma 1.2. *Let ω be a nonnegative, decomposable, normalized element of $\Lambda^k(\mathbb{R}^n)$ with $k < n$. Then there is $\eta \in \Lambda^{k+1}(\mathbb{R}^n)$, nonnegative and nonzero with $\omega \subset \eta$. If ω is positive, then η may be chosen to be positive.*

Proof. Let

$$\omega = \sum_A \omega_A e_A.$$

Let j be the smallest integer for which it is not the case that $k \in A$ for every $k \leq j$ and $\omega_A \neq 0$. Then

$$\eta = (-1)^{j-1} e_j \wedge \omega,$$

is nonnegative. This proves the first part of the lemma.

To prove the second part of the lemma, we proceed by induction on n. If $n = k+1$ then we simply observe that

$$\omega \subset e_{\{1,\ldots,n\}}.$$

Now, for general n, we write

$$\omega = \omega_1 + \omega_2,$$

with

$$\omega_1 = \sum_{1 \in A} \omega_A e_A \quad \text{and} \quad \omega_2 = \sum_{1 \notin A} \omega_A e_A.$$

Now by the induction hypothesis, we can find v orthogonal to e_1 so that

$$\mu = v \wedge \omega_2 = \sum_{1 \notin A} \mu_A e_A,$$

has the property that all μ_A with $1 \notin A$ are strictly positive.

We fix $\epsilon > 0$ to be determined later. We let

$$\eta_\epsilon = (e_1 + \epsilon v) \wedge (\omega_1 + \omega_2) = e_1 \wedge \omega_2 + \epsilon v \wedge \omega_1 + \epsilon v \wedge \omega_2.$$

Observe that the third term is the only one which has components not involving e_1 and that by assumption those terms are all strictly positive. We now pick ϵ sufficiently small so that the components of $e_1 \wedge \omega_2$ dominate the components of $\epsilon v \wedge \omega_1$. Thus η_ϵ is positive and we may choose $\eta = \eta_\epsilon$. $\square$

Remark:. Note the duality between the above proofs. In fact, the map from subsets A to complementary subsets $A^{\complement}$ induces an automorphism of $\Lambda(\mathbb{R}^n)$ taking $\omega \in \Lambda^k(\mathbb{R}^n)$ to $\omega^{\complement} \in \Lambda^{n-k}(\mathbb{R}^n)$ which respects positivity. The equation $\omega \wedge (\eta^{\complement}) = Q(\omega, \eta)\, e_1 \wedge e_2 \wedge \cdots \wedge e_n$ defines a nondegenerate quadratic form Q on $\Lambda^k(\mathbb{R}^n)$ which can be viewed as a quadratic form on $\mathbb{R}^n$ when $k = 1$. The map $G(k,n) \to G(n-k,n)$ given by $V \mapsto V^{\perp_Q}$ gives the desired (inclusion reversing) duality between decomposable forms relating Lemma 1.2 to Lemma 1.1.

§2 Topological lemmas

We proceed to state the main lemma.

Lemma 2.1. *Let $Q = [0,1] \times [-1,1]^{n-1}$. We denote points of Q by (t,x) with $t \in [0,1]$ and $x \in [-1,1]^{n-1}$. Let $\mathcal{V}$ be an m-dimensional vector bundle (of course, trivial) on Q. (We consider $\mathcal{V}$ as embedded in $\mathbb{R}^{n+m}$.) Let $H_1, \ldots, H_N$ be closed half-space sections in the bundle of half-spaces of fibers of $\mathcal{V}$. Suppose that for each $p \in Q$ which is either in the interior of Q or of the form $p = (0,x)$ with x in the interior of $[-1,1]^{n-1}$, we have that $H_1(p) \cap H_2(p) \cap \cdots \cap H_N(p)$ is bounded and has nonempty interior in the fiber over p. Then there is a homeomorphism ϕ from*

$$E = \bigcup_{p \in [0,1]\times[-1,1]^{n-1}} H_1(p) \cap H_2(p) \cap \cdots \cap H_N(p),$$

to the closed half ball HB in $\mathbb{R}^{n+m} = \mathbb{R} \times \mathbb{R}^{n+m-1}$, i.e.

$$HB = \{(t,x) \in \mathbb{R} \times \mathbb{R}^{n+m-1} : t \geq 0; |t|^2 + |x|^2 \leq 1\},$$

so that if we define the bottom EB of E by

$$EB = \bigcup_{x \in [-1,1]^{n-1}} H_1(0,x) \cap H_2(0,x) \cap \cdots \cap H_N(0,x),$$

and we define the bottom of the half ball HBB by

$$HBB = \{(0,x) \in \mathbb{R} \times \mathbb{R}^{n+m-1} : |x| \leq 1\},$$

then

$$\phi(EB) = HBB.$$

Proof. For convenience, in what follows we will denote the fiber over the point $p \in Q$ by $E(p)$. By hypothesis, we have that the origin $0 \in \mathbb{R}^n$ is contained in the interior of the bottom QB of Q, namely

$$QB = \{0\} \times [-1,1]^{n-1}.$$

Consider the ordinary barycenter of the fiber over $p \in Q$, $b(p) = \frac{1}{|E(p)|} \int_{E(p)} y dy$. Since this varies continuously on Q, the map $(p,y) \longrightarrow (p, y - b(p))$ is a homeomorphism of E onto its image, which preserves fibers. Henceforth we will identify E with its image and assume that in this way each fiber has been "centered" along the 0 section.

We introduce the distinguished boundary dQ of Q, where

$$dQ = \partial Q \backslash (\{0\} \times (-1,1)^{n-1}).$$

Notice that only for $p \in dQ$ do the fibers $Q(p)$ fail to have nonempty interior.

Let $P : Q\backslash\{0\} \longrightarrow dQ$ be the radial projection map. (That is if $p \in Q\backslash\{0\}$ then $P(p)$ is the unique point of dQ contained in the ray starting at 0 and containing p.) Let $E' \subset \mathbb{R}^{n+m}$ denote the union of all line segments connecting every point of $E(0)$ to every point of $E(q)$ for all $q \in dQ$. Explicitly,

$$E' = \bigcup_{q \in dQ} \bigcup_{y \in E(q)} \bigcup_{z \in E(0)} [y, z],$$

where $[y, z]$ is the closed line segment from y to z.

We claim that E is homeomorphic to E' and that E' is star convex from the origin in $\mathbb{R}^{n+m}$. For the first claim we observe that for each $p \in Q$ the fiber $E'(p)$ is also convex with barycenter $0 \in \mathbb{R}^m$. This follows from the fact that the join of the two convex sets $E(P(p))$ and $E(0)$ is again convex, and the intersection of this convex set with the convex set $(p, \mathbb{R}^m)$ is again convex. Since the barycenter of the join sets is 0, so is each slice. Therefore the homeomorphism from E to E' is given by a radial rescaling projection from the point $(p, 0)$ in each fiber. (The fiber-wise homeomorphisms depend continuously on p and are the identity at $p = 0$ and $p \in dQ$.)

Now we claim that E' is star convex from 0. To see this, let $v(t)$ denote the unit speed linear ray emanating from $0 \in \mathbb{R}^{n+m}$ in the direction of the unit vector v. Suppose $v(t)$ first exits E' at a point (x, y) on the boundary of E'. If the ray enters E' again it must do so in the portion of E' lying over the ray in $Q \subset \mathbb{R}^n$ which is the projection of $v(t)$ to $\mathbb{R}^n$. However this is impossible since this set is the join of $E(P(c(t)))$ and $E(0)$, and hence a convex set in $\mathbb{R}^{n+k}$. Moreover the time of exit, say T_v, for the ray $v(t)$ depends continuously on the direction v.

Now the explicit map $v(t) \longrightarrow \frac{v(t)}{T_v}$ for $t \leq T_v$ and all v in the closed unit half sphere sphere, is a homeomorphism of E' onto the closed unit half ball HB which maps the bottom of E', namely $\cup_{t \in [-1,1]^{n-1}} E'(0, t)$ onto the bottom of the half ball HBB. $\square$

We refer to a body E obtained as in the proof of Lemma 2.1 as an $n+m$ convexoid and we refer EB as its bottom. Note that in this definition, we forget the values of n and m and retain only the dimension $n + m$.

Corollary 2.2. *Let E and F be l convexoids and let EB and FB their bottoms. Let ϕ be a homeomorphism from EB onto FB and let X be the topological space obtained from $E \cup F$ with the bottoms EB and FB identified by ϕ. Then X is homeomorphic to a closed l ball.*

Proof. First observe that there is a homeomorphism from a closed l-dimensional half ball HB to a cylinder $[0, 1] \times B^{l-1}$ which maps the bottom HBB to the base of the cylinder $\{0\} \times B^{l-1}$. Thus by lemma 2.1, there is a homeomorphism from E to $[0, 1] \times B^{l-1}$ which sends EB to $\{1\} \times B^{l-1}$ and a homeomorphism from F to $[1, 2] \times B^{l-1}$ which sends FB

to $\{1\} \times B^{l-1}$. Gluing the two cylinders by the homeomorphism induced from ϕ, we see that $E \cup F$ is homeomorphic to a cylinder and hence to a ball. $\square$

In the following section, we will prove that the set of nonnegative elements of a Grassmannian is homeomorphic to a ball by decomposing this set into two convexoids glued at their bottoms.

§3 Proof of the main theorem

We let $G(k, n)$ be the Grassmannian of k-planes in $\mathbb{R}^n$ containing 0. To any such k-plane P, there corresponds a decomposable k-vector, unique up to a constant, which can be found as the wedge of k linearly independent vectors in P. We fix a basis $e_1, \dots, e_n$ for $\mathbb{R}^n$ and define the inner product which makes this basis orthonormal. We say that a plane P is positive (resp. nonnegative) if it has a corresponding k-vector which is positive (resp. nonnegative). We denote the positive (resp. nonnegative) elements of $G(k, n)$ as $G(k, n)_+$ (resp. $G(k, n)_{\geq 0}$). To each nonnegative k-plane corresponds a unique nonnegative, normalized, decomposable k-vector and this correspondence is a homeomorphism.

Theorem 3.1. *The set $G(k, n)_{\geq 0}$ viewed as a closed subset of $G(k, n)$ is homeomorphic to a closed ball.*

Proof. We will proceed by double induction on n and k. Note that since every 1-vector is decomposable, we have that $G(1, n)_{\geq 0}$ is homeomorphic to a closed simplex and therefore the theorem is trivial in that case. Similarly, since every $n-1$ vector is decomposable, we have that $G(n-1, n)_{\geq 0}$ is also homeomorphic to a ball. We shall prove that if we know that $G(k, n-1)_{\geq 0}$ and $G(k-1, n-1) \geq 0$ are both homeomorphic to balls then $G(k, n)_{\geq 0}$ is homeomorphic to a ball. This suffices to prove the Theorem.

Our first step will be to cleverly parametrize $G(k, n)_{\geq 0}$. Any nonnegative, normalized, decomposable k-vector ρ can be written either as

$$\rho = (e_1 + v) \wedge \eta_0,$$

where v is a vector in the span of $e_2, \dots, e_n$ and η_0 is a nonnegative, decomposable $k-1$-vector involving only $e_2, \dots, e_n$, or as

$$\rho = \omega,$$

where

$$\omega = \sum_{1 \notin A} \omega_A e_A,$$

with ω nonnegative, normalized, and decomposable. Note further that in the first case, $v \wedge \eta_0$ is nonnegative and decomposable. Let t be the sum of the components of η_0. Then if t is nonzero, we define

$$\eta = \frac{1}{t} \eta_0,$$

and since $(1-t)$ is nonzero, we define

$$\omega = \frac{1}{1-t}(v \wedge \eta_0).$$

(Here, we intentionally defined a k-vector as ω in both cases. Note that ω depends continuously on ρ as long as $t \neq 0$. Moreover, we define $t = 0$ in the second case and see that t varies continuously with ρ. We have that when $t \neq 0$, then η is a nonnegative, normalized, decomposable $k-1$-vector involving only $e_2, \dots, e_n$ and when $t \neq 1$ then ω is a nonnegative, normalized, decomposable k-vector involving only $e_2, \dots, e_n$ with $\eta \subset \omega$. Conversely given the triple t, η, ω, we can reconstruct ρ as

$$\rho = te_1 \wedge \eta + (1-t)\omega.$$

Thus we have a kind of parametrization for $G(k,n)_{\geq 0}$ which degenerates at $t = 0$ and $t = 1$. We break $G(k,n)_{\geq 0}$ into two pieces, E, the set where $t \leq \frac{1}{2}$ and F, the set where $t \geq \frac{1}{2}$.

We consider F first. We view it as a fibration over pairs $(t,\eta) \in [\frac{1}{2}, 1] \times G(k-1, n-1)_{\geq 0}$. If $t = 1$, then the fiber degenerates to a point. If $t \neq 1$, then the fiber consists of the set of all $\omega \in G(k,n)_{\geq 0}$ with $\eta \subset \omega$. Any decomposable k form which contains η is the wedge of η with a vector in the orthogonal complement of the plane associated to η. Thus there is an $n-k+1$ dimensional vector space of decomposable k-vectors containing η. The normalized decomposable k-vectors containing η are a codimension 1 affine subspace (i.e having dimension $n-k$.) The set of all nonnegative, normalized, decomposable k-vectors which contain η is the intersection of the $n-k$-dimensional affine subspace with the simplex of all nonnegative normalized k-vectors. Therefore the fiber is a convex polytope of dimension at most $n-k$. Applying Lemma 1.2, we see it is nonempty for any nonnegative η and that for any positive η, we can find a positive ω, so that by perturbing, we see that we have an $n-k$ dimensional convex polytope with nonempty interior. (And indeed by construction, these polytopes vary continuously with the base and shrink to points as t tends to 1.) To sum up, F is a fibration over the base space $[\frac{1}{2}, 1] \times G(k-1, n-1)_{\geq 0}$. By the induction hypothesis, $G(k-1, n-1)_{\geq 0}$ is homeomorphic to a ball and hence a cube. We have shown the fiber is always a convex polytope in an $n-k$ dimensional vector space. (Since the base is homeomorphic to a ball, we know that the bundle of these vector spaces is trivial.) Moreover, we know that the fiber has nonempty interior, whenever $t \neq 1$ and η is positive (in other words, in the interior of $G(k-1, n-1)_{\geq 0}$.) Thus F is a convexoid and the bottom FB is the part of the fibration over $\{\frac{1}{2}\} \times G(k-1, n-1)_{\geq 0}$.

Now we consider E. We can view it as a fibration over $[0, \frac{1}{2}] \times G(k, n-1)_{\geq 0}$. Again by the induction hypothesis, we have that $G(k, n-1)_{\geq 0}$ is homeomorphic to a ball and hence to a cube. Now we must consider the fiber. When $t = 0$, it degenerates to a point. Otherwise, for a given nonnegative, normalized, decomposable k-vector ω, it is the set of

nonnegative,normalized,decomposable $k-1$-vectors η with $\eta \subset \omega$. Observing that the set of $\eta \subset \omega$ of codimension 1, may be identified with the set of vectors v in the k plane corresponding to ω (by orthogonal complementation), and that nonnegativity is a convex condition, we see that the fiber of E is a convex polytope of dimension at most $k-1$. Further, applying Lemma 1.1, we see that the fiber has nonempty interior whenever $t \neq 0$ and ω positive. Thus E is a convexoid and its bottom EB is the part of the fibration over $\{\frac{1}{2}\} \times G(k, n-1)_{\geq 0}$. Noticing that $EB = FB$, we apply corollary 2.2 to see that $E \cup F$ is homeomorphic to a ball. Thus we have proved the theorem. $\square$

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